Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 617–640
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Kähler metrics generated by functions of the time-like distance in the flat Kähler–Lorentz space

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Received 26 October 2005; received in revised form 16 May 2006; accepted 17 May 2006
Available online 3 July 2006

Abstract

We prove that every Kähler metric, whose potential is a function of the time-like distance in the flat Kähler–Lorentz space, is of quasi-constant holomorphic sectional curvatures, satisfying certain conditions. This gives a local classification of the Kähler manifolds with the above-mentioned metrics. New examples of Sasakian space forms are obtained as real hypersurfaces of a Kähler space form with special invariant distribution. We introduce three types of even dimensional rotational hypersurfaces in flat spaces and endow them with locally conformal Kähler structures. We prove that these rotational hypersurfaces carry Kähler metrics of quasi-constant holomorphic sectional curvatures satisfying some conditions, corresponding to the type of the hypersurfaces. The meridians of those rotational hypersurfaces, whose Kähler metrics are Bochner–Kähler (especially of constant holomorphic sectional curvatures), are also described.

MSC: primary 53B35; secondary 53B30

Keywords: Kähler manifolds with J-invariant distributions; Kähler manifolds of quasi-constant holomorphic sectional curvatures; Kähler–Lorentz manifolds; Rotational hypersurfaces with complex structure

1. Introduction

In [3] we have given a complete description of the curvature tensor and curvature properties of the Kähler metrics $g = \partial\bar{\partial} f(r^2)$, where r^2 is the distance function with respect to the origin in $\mathbb{C}^n$ and the real $\mathcal{C}^\infty$-function $f(r^2)$ satisfies the conditions

$$f'(r^2) > 0, \quad f'(r^2) + r^2 f''(r^2) > 0.$$

Bochner–Kähler metrics of the type $\partial\bar{\partial} f(r^2)$ have been studied in [6]. The completeness of these metrics has been discussed in [1].

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.05.004

We have introduced the notion of a Kähler manifold (M, g, J, D) $(\dim M = 2n \geqq 6)$ with J-invariant B_0-distribution D $(\dim D = 2(n-1))$. Any B_0-distribution generates a function $k > 0$ on M. If $D^{\perp}$ is the distribution, orthogonal to D, then every holomorphic section $E(p),\ p \in M$, determines a geometric angle $\vartheta = \angle(E(p), D^{\perp}(p))$.

A Kähler manifold (M, g, J, D) is of quasi-constant holomorphic sectional curvatures if its holomorphic sectional curvatures only depend on the point p and the angle ϑ.

If (M, g, J, D) is a Kähler manifold of quasi-constant holomorphic sectional curvatures, then the distribution $D(p),\ p \in M$ is of pointwise constant holomorphic sectional curvatures $a(p)$ and the function $a + k^2$ divides the class of these manifolds into three subclasses according to

$$a + k^2 > 0, \qquad a + k^2 = 0, \qquad a + k^2 < 0.$$

In [3] we have shown that the flat Kähler manifold $\mathbb{C}^n$ carries a canonical B_0-distribution and proved the following characterization of the family of Kähler metrics $g = \partial\bar{\partial} f(r^2)$:

Any Kähler metric $g = \partial\bar{\partial} f(r^2)$ is of quasi-constant holomorphic sectional curvatures with $a + k^2 > 0$.

Conversely, any Kähler manifold M $(\dim M = 2n \geq 6)$ of quasi-constant holomorphic sectional curvatures with B_0-distribution and $a + k^2 > 0$ is locally equivalent to $(\mathbb{C}^n, g, J_0)$ with the canonical B_0-distribution and $g = \partial\bar{\partial} f(r^2)$.

In this paper we solve the problem of describing the curvature properties of the Kähler metrics generated by potential functions $f(-r^2)$, $-r^2$ being the time-like distance function from the origin in the flat Kähler–Lorentz space.

Let $(\mathbb{C}^n, h', J_0)$ be the flat Kähler–Lorentz space with the canonical complex structure J_0 and flat Kähler metric h' of signature $(2(n-1), 2)$.

In Proposition 3.5 we prove that if $f(-r^2)$, $-r^2 < 0$, is a real $\mathcal{C}^{\infty}$-function satisfying the conditions

$$f'(-r^2) > 0, \quad f'(-r^2) - r^2 f''(-r^2) < 0,$$

then $g = \partial\bar{\partial} f(-r^2)$ is a positive definite Kähler metric in the time-like domain $\mathbb{T}_1^{n-1} = \{\mathbf{Z} \in \mathbb{C}^n : h'(\mathbf{Z}, \mathbf{Z}) < 0\}$.

In Section 4 we prove the basic Theorem 4.7, which gives a complete curvature description of the family of Kähler metrics $g = \partial\bar{\partial} f(-r^2)$:

Any Kähler metric $g = \partial\bar{\partial} f(-r^2)$ is of quasi-constant holomorphic sectional curvatures with $a + k^2 < 0$.

Conversely, every Kähler manifold M $(\dim M = 2n \geq 6)$ of quasi-constant holomorphic sectional curvatures with B_0-distribution and $a + k^2 < 0$ is locally equivalent to $(\mathbb{T}_1^{n-1}, g, J_0)$ with the canonical B_0-distribution and $g = \partial\bar{\partial} f(-r^2)$.

In Section 5 we clear up the geometric meaning of the function $a + k^2$ in a Kähler manifold (M, g, J, D) of quasi-constant holomorphic sectional curvatures. We show that (M, g, J, D) is a one-parameter family of α-Sasakian space forms $Q^{2n-1}(s),\ s \in I$ with $\alpha = \frac{k}{2}$ and prove in Proposition 5.1 that $\operatorname{sign}(a + k^2)$ determines the type of the corresponding $Q^{2n-1}(s)$.

As a consequence of Theorem 4.7 we obtain examples of Kähler space forms in $\mathbb{T}_1^{n-1}$ with B_0-distribution and $a + k^2 < 0$. In particular the metric $g = -2\partial\bar{\partial} \ln(r^2 - 1), \quad -r^2 < -1$, is of constant holomorphic sectional curvature -1. Considering the unit "disc" $(\mathbb{D}_1^{n-1}(1) : h'(\mathbf{Z}, \mathbf{Z}) < -1)$ we show that any hypersphere $H_1^{2n-1}(O, r),\ r > 1$ in $(\mathbb{D}_1^{n-1}, g, J_0)$ carries a natural structure of an α-Sasakian space form with $\alpha = \frac{1}{2r}$ and constant φ-holomorphic sectional curvatures c, so that $c + 3\alpha^2 < 0$ (cf. [7]).

In Section 6 we consider three types of rotational hypersurfaces M in $\mathbb{C}^n \times \mathbb{R}$ with axis of revolution $l = \mathbb{R}$:

Type I: the parallels S^{2n-1} are the usual hyperspheres in the complex Euclidean space $(\mathbb{C}^n, g', J_0)$ and the axis of revolution $\mathbb{R}$ is endowed with positive definite inner product; the meridians are curves in the Euclidean plane.

Type II: the parallels S^{2n-1} are the usual hyperspheres in the complex Euclidean space $(\mathbb{C}^n, g', J_0)$ and the axis $\mathbb{R}$ is endowed with negative definite inner product; the meridians are space-like curves in the hyperbolic plane.

Type III: the parallels H_1^{2n} are hyperspheres in the flat time-like domain $(\mathbb{T}_1^{n-1}, h', J_0)$ and the axis $\mathbb{R}$ is endowed with positive definite inner product; the meridians are time-like curves in the hyperbolic plane.

In Section 6.1 we recall that the hypersurfaces of type I carry a natural Kähler structure of quasi-constant holomorphic sectional curvatures with functions $a > 0$, $a + k^2 > 0$. In Proposition 6.3 we obtain the meridians of the rotational hypersurfaces of type I, whose Kähler metric is Bochner–Kähler.

In Section 6.2 we introduce a Kähler structure on rotational hypersurfaces of type II and prove in Theorem 6.6 that this Kähler structure is of quasi-constant holomorphic sectional curvatures with functions $a < 0, a + k^2 > 0$. We find the meridians of the rotational hypersurfaces of type II, whose Kähler metric is Bochner–Kähler (Proposition 6.8) or of constant holomorphic sectional curvatures (Proposition 6.7).

In Section 6.3 we introduce a Kähler structure on the rotational hypersurfaces of type III and prove in Theorem 6.11 that this Kähler structure is of quasi-constant holomorphic sectional curvatures with functions $a < 0, a + k^2 < 0$. We find the meridians of those rotational hypersurfaces of type III, whose Kähler metric is Bochner–Kähler (Proposition 6.13) or is of constant holomorphic sectional curvatures (Proposition 6.12).

2. Preliminaries

In this section we give some basic notions and formulas for Kähler manifolds with B_0-distribution [3] we need further.

Let (M, g, J, D) be a $2n$-dimensional Kähler manifold with metric g, complex structure J and J-invariant distribution D of codimension 2. The Lie algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}M$ and T_pM will stand for the tangent space to M at any point $p \in M$. In the presence of the distribution D the structure of any tangent space is $T_pM = D(p) \oplus D^\perp(p)$, where $D^\perp(p)$ is the two-dimensional J-invariant orthogonal complement to the space $D(p)$. This means that the structural group of the manifolds under consideration is the subgroup $U(n-1) \times U(1)$ of $U(n)$.

In the local treatment of these manifolds $D^\perp = \text{span}\{\xi, J\xi\}$ for some unit vector field ξ. The 1-forms, corresponding to ξ and $J\xi$, respectively, are

$$\eta(X) = g(\xi, X), \qquad \tilde{\eta}(X) = g(J\xi, X) = -\eta(JX); \quad X \in \mathfrak{X}M.$$

Then the distribution D is determined by the conditions

$$D(p) = \{X \in T_pM \mid \eta(X) = \tilde{\eta}(X) = 0\}, \quad p \in M.$$

The Kähler form Ω of the structure (g, J) is given by $\Omega(X, Y) = g(JX, Y)$, $X, Y \in \mathfrak{X}M$.

Let ∇ be the Levi-Civita connection of the metric g.

A J-invariant distribution D, $(D^\perp = \text{span}\{\xi, J\xi\})$ is said to be a B_0-distribution [3] if

(i) $\nabla_{x_0}\xi = \frac{k}{2}x_0, \quad k \neq 0, x_0 \in D;$

(ii) $\nabla_{J\xi}\xi = -p^*J\xi;$

(iii) $\nabla_\xi\xi = 0.$

The above definition implies immediately the following equalities [3]:

$$\nabla_X\xi = \frac{k}{2}\{X - \eta(X)\xi - \tilde{\eta}(X)J\xi\} - p^*\tilde{\eta}(X)J\xi, \quad X \in \mathfrak{X}M; \tag{2.1}$$

$$\mathrm{d}k = \xi(k)\,\eta, \qquad p^* = -\frac{\xi(k) + k^2}{k}. \tag{2.2}$$

Any Kähler manifold (M, g, J, D) with J-invariant distribution D carries the tensors

$$4\pi(X, Y)Z := g(Y, Z)X - g(X, Z)Y - 2g(JX, Y)JZ + g(JY, Z)JX - g(JX, Z)JY; \tag{2.3}$$

$$\begin{aligned}
\Phi_1(X, Y)Z &:= \frac{1}{8}\{g(Y, Z)(\eta(X)\xi + \tilde{\eta}(X)J\xi) - g(X, Z)(\eta(Y)\xi + \tilde{\eta}(Y)J\xi) \\
&\quad + g(JY, Z)(\eta(X)J\xi - \tilde{\eta}(X)\xi) - g(JX, Z)(\eta(Y)J\xi - \tilde{\eta}(Y)\xi) \\
&\quad - 2g(JX, Y)(\eta(Z)J\xi - \tilde{\eta}(Z)\xi)\}; \\
\Phi_2(X, Y)Z &:= \frac{1}{8}\{(\eta(Y)\eta(Z) + \tilde{\eta}(Y)\tilde{\eta}(Z))X - (\eta(X)\eta(Z) + \tilde{\eta}(X)\tilde{\eta}(Z))Y \\
&\quad + (\eta(Y)\tilde{\eta}(Z) - \tilde{\eta}(Y)\eta(Z))JX - (\eta(X)\tilde{\eta}(Z) - \tilde{\eta}(X)\eta(Z))JY \\
&\quad - 2(\eta(X)\tilde{\eta}(Y) - \tilde{\eta}(X)\eta(Y))JZ\};
\end{aligned} \tag{2.4}$$

$$\Phi := \Phi_1 + \Phi_2;$$
$$\begin{aligned}\Psi(X,Y)Z &:= \eta(Y)\eta(Z)\tilde{\eta}(X)J\xi - \eta(X)\eta(Z)\tilde{\eta}(Y)J\xi + \eta(X)\tilde{\eta}(Y)\tilde{\eta}(Z)\xi - \eta(Y)\tilde{\eta}(X)\tilde{\eta}(Z)\xi \\ &= (\eta\wedge\tilde{\eta})(X,Y)(\tilde{\eta}(Z)\xi - \eta(Z)J\xi),\end{aligned} \tag{2.5}$$

$X, Y, Z \in \mathfrak{X}M$. These tensors are invariant under the action of the structural group $U(n-1)\times U(1)$ [10].

The Riemannian curvature tensor R of the metric g is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$
$$R(X,Y,Z,U) = g(R(X,Y)Z,U); \quad X,Y,Z,U \in \mathfrak{X}M.$$

In [3] we proved that a Kähler manifold (M, g, J, D) ($\dim M = 2n \geq 4$) with J-invariant distribution D is of quasi-constant holomorphic sectional curvatures if and only if

$$R = a\pi + b\Phi + c\Psi,$$

where a, b and c are functions on M, generated by the structure (g, J, ξ).

If (M, g, J, D) ($\dim M = 2n \geq 6$) is a Kähler manifold of quasi-constant holomorphic sectional curvatures, then the following statements hold good [3]:

(i) If D is a B_0-distribution, then

$$\mathrm{d}a = \frac{kb}{2}\eta. \tag{2.6}$$

(ii) Under the condition $b \neq 0$, D is a B_0-distribution if and only if D is non-involutive.

(iii) If $b = 0$ and D is non-involutive, then $c = 0$, i.e. M is a Kähler space form.

Finally we recall some basic facts related to α-Sasakian manifolds.

Let $Q^{2n-1}(g, \varphi, \tilde{\xi}, \tilde{\eta})$ ($n \geq 3$) be an almost contact Riemannian manifold, i.e.

$$\begin{aligned}&g(\varphi x, \varphi y) = g(x,y) - \tilde{\eta}(x)\tilde{\eta}(y), \quad x, y \in \mathfrak{X}Q^{2n-1},\\ &\varphi^2 x = -x + \tilde{\eta}(x)\tilde{\xi}, \quad x \in \mathfrak{X}Q^{2n-1},\\ &\varphi\xi = 0.\end{aligned} \tag{2.7}$$

If the structure $(g, \varphi, \tilde{\xi}, \tilde{\eta})$ of an almost contact Riemannian manifold Q^{2n-1} satisfies the conditions

$$\mathcal{D}_x\tilde{\xi} = \alpha\,\varphi x, \quad x \in \mathfrak{X}Q^{2n-1},$$
$$(\mathcal{D}_x\varphi)(y) = \alpha\,(\tilde{\eta}(y)x - g(x,y)\tilde{\xi}), \quad x, y \in \mathfrak{X}Q^{2n-1},$$

where $\mathcal{D}$ is the Levi-Civita connection of the metric g and $\alpha = \text{const}$, then Q^{2n-1} is called an α-*Sasakian manifold* [4].

If the constant $\alpha = 1$, then Q^{2n-1} is a Sasakian manifold in the usual sense.

α-Sasakian space forms are characterized as follows:

Proposition 2.1 ([5,4]). *An α-Sasakian manifold* $(Q^{2n-1}, g, \varphi, \tilde{\xi}, \tilde{\eta})$ ($\dim Q^{2n-1} \geq 5$) *is of constant φ-holomorphic sectional curvatures c if and only if*

$$\begin{aligned}K(x,y,z,u) = {}&\frac{c+3\alpha^2}{4}[g(y,z)g(x,u) - g(x,z)g(y,u)]\\ &+\frac{c-\alpha^2}{4}[g(\varphi y,z)g(\varphi x,u) - g(\varphi x,z)g(\varphi y,u) - 2g(\varphi x,y)g(\varphi z,u)\\ &-g(y,z)\tilde{\eta}(x)\tilde{\eta}(u) - g(x,u)\tilde{\eta}(y)\tilde{\eta}(z)\\ &+g(x,z)\tilde{\eta}(y)\tilde{\eta}(u) + g(y,u)\tilde{\eta}(x)\tilde{\eta}(z)], \quad x,y,z,u \in \mathfrak{X}Q^{2n-1}.\end{aligned}$$

We note that there are three types of α-Sasakian space forms with respect to $\mathrm{sign}(c+3\alpha^2)$ [7]:

Type I: $c + 3\alpha^2 > 0$;

Type II: $c + 3\alpha^2 = 0$;

Type III: $c + 3\alpha^2 < 0$.

3. Kähler–Lorentz manifolds with B_0-distributions

Let (M, h', J) ($\dim M = 2n$) be a complex manifold with complex structure J and indefinite Hermitian metric h' of signature $(2(n-1), 2)$ and ∇' be the Levi-Civita connection of h'. If $\nabla' J = 0$, then (M, h', J) is said to be a *Kähler–Lorentz manifold*.

We consider Kähler–Lorentz manifolds (M, h', J) with a space-like J-invariant distribution D of $\dim D = 2(n-1)$. Then the orthogonal J-invariant two-dimensional distribution $D^\perp$ is time-like.

Since our considerations are local, we can assume the existence of a time-like unit vector field ξ' on M such that $D^\perp(p) = \operatorname{span}\{\xi', J\xi'\}$ at any point $p \in M$. We denote by η' and $\tilde{\eta}'$ the unit 1-forms corresponding to ξ' and $J\xi'$, respectively, i.e.

$$\eta'(X) = h'(\xi', X), \qquad \tilde{\eta}'(X) = h'(J\xi', X) = -\eta'(JX), \quad X \in \mathfrak{X}M;$$
$$\|\eta'\|^2 = \|\tilde{\eta}'\|^2 = \eta'(\xi') = \tilde{\eta}'(J\xi') = -1.$$

Then the space-like distribution D is determined by the conditions

$$D(p) = \{X \in T_pM \mid \eta'(X) = \tilde{\eta}'(X) = 0\}, \quad p \in M.$$

The Riemannian curvature tensor R' of ∇' is determined as in the previous section. We note that the Ricci tensor ρ' and the scalar curvature τ' of the metric h' are given by

$$\rho'(Y, Z) = \sum_{i=1}^{2n} h'(e_i, e_i) R'(e_i, Y, Z, e_i), \quad Y, Z \in \mathfrak{X}M;$$
$$\tau' = \sum_{i=1}^{2n} h'(e_i, e_i) \rho'(e_i, e_i),$$

where $\{e_i\}$, $i = 1, \ldots, 2n$ is an orthonormal basis for T_pM, $p \in M$.

We also note that the tensor $h'^\perp = -(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')$ does not depend on the basis $\{\xi', J\xi'\}$ of $D^\perp$. This tensor is negative definite and it is the restriction of the metric h' onto the distribution $D^\perp$.

The Kähler form Θ of the structure (h', J) is given by $\Theta(X, Y) = h'(JX, Y)$, $X, Y \in \mathfrak{X}M$.

All directions in $D^\perp = \operatorname{span}\{\xi', J\xi'\}$ have one and the same Ricci curvature, which is denoted by σ', i.e.

$$\sigma' = -\rho'(\xi', \xi') = -\rho'(J\xi', J\xi'). \tag{3.1}$$

The Riemannian sectional curvature of the distribution $D^\perp$ is denoted by $\varkappa'$, i.e.

$$\varkappa' = R'(\xi', J\xi', J\xi', \xi'). \tag{3.2}$$

Thus the structure (h', J, D) gives rise to the functions $\varkappa'$, σ' and τ'.

Any vector field $X \in \mathfrak{X}M$ is decomposable in a unique way as follows:

$$X = x_0 - \tilde{\eta}'(X)J\xi' - \eta'(X)\xi',$$

where x_0 is the projection of X into $\mathfrak{X}D$.

As a rule, we use the following denotations for vector fields (vectors):

$$X, Y, Z \in \mathfrak{X}M \ (T_pM); \quad x_0, y_0, z_0 \in \mathfrak{X}D \ (D(p)).$$

If $D^\perp = \operatorname{span}\{\xi', J\xi'\}$, then the relative divergences $\operatorname{div}_0\xi'$ and $\operatorname{div}_0 J\xi'$ (the relative codifferentials $\delta_0\eta'$ and $\delta_0\tilde{\eta}'$) of the vector fields ξ' and $J\xi'$ (of the 1-forms η' and $\tilde{\eta}'$) with respect to the space-like distribution D are introduced as in the definite case:

$$\operatorname{div}_0\xi' = -\delta_0\eta' = \sum_{i=1}^{2(n-1)} (\nabla'_{e_i}\eta')e_i, \qquad \operatorname{div}_0 J\xi' = -\delta_0\tilde{\eta}' = \sum_{i=1}^{2(n-1)} (\nabla'_{e_i}\tilde{\eta}')e_i,$$

where $\{e_1, \ldots, e_{2(n-1)}\}$ is an orthonormal basis of $D(p)$, $p \in M$.

The restriction of the metric h' onto the distribution Δ

$$\Delta(p) := \{X \in T_pM \mid \eta'(X) = 0\}, \quad p \in M,$$

perpendicular to ξ', is of signature $(2(n-1), 1)$.

The notion of a space-like B_0-distribution in a Kähler–Lorentz manifold is introduced similarly to the definite case:

Definition 3.1. Let (M, h', J, D) $(\dim M = 2n \geq 6)$ be a Kähler–Lorentz manifold with J-invariant space-like distribution D $(D^\perp = \operatorname{span}\{\xi', J\xi'\})$. The distribution D is said to be a B_0-*distribution* if:

$$\begin{aligned} &\text{(i)} \quad \nabla'_{x_0}\xi' = -\frac{k'}{2}x_0, \quad k' \neq 0, x_0 \in D; \\ &\text{(ii)} \quad \nabla'_{J\xi'}\xi' = p^{*\prime}J\xi'; \\ &\text{(iii)} \quad \nabla'_{\xi'}\xi' = 0, \end{aligned} \tag{3.3}$$

where k' and $p^{*\prime}$ are functions on M.

Next we prove some properties of Kähler–Lorentz manifolds with B_0-distribution.

Lemma 3.2. *Let* (M, h', J, D) $(\dim M = 2n \geq 6)$ *be a Kähler–Lorentz manifold with* B_0*-distribution* D $(D^\perp = \operatorname{span}\{\xi', J\xi'\})$*. Then*

$$\mathrm{d}k' = -\xi'(k')\eta', \qquad p^{*\prime} = \frac{\xi'(k') - k'^2}{k'}.$$

Proof. The conditions (3.3) imply

$$\nabla'_X\xi' = -\frac{1}{2}k'\{X + \tilde\eta'(X)J\xi' + \eta'(X)\xi'\} - p^{*\prime}\tilde\eta'(X)J\xi'. \tag{3.4}$$

By using (3.4) we find $\mathrm{d}\tilde\eta'$ and after an exterior differentiation we obtain the assertion of the lemma. □

Lemma 3.3. *Let* (M, h', J, D) $(\dim M = 2n \geq 6)$ *be a Kähler–Lorentz manifold with* B_0*-distribution* D $(D^\perp = \operatorname{span}\{\xi', J\xi'\})$*. Then*

$$\begin{aligned} R'(X, Y)\xi' = {} & \frac{1}{2}\left(\xi'(k') - \frac{1}{2}k'^2\right)\{\eta'(X)Y - \eta'(Y)X + 2h'(JX, Y)J\xi' - \tilde\eta'(X)JY + \tilde\eta'(Y)JX\} \\ & - \frac{1}{k'}\,\xi'\left(\xi'(k') - \frac{1}{2}k'^2\right)(\eta' \wedge \tilde\eta')(X, Y)J\xi'; \end{aligned} \tag{3.5}$$

$$\varkappa' = -\frac{1}{k'}\,\xi'\left(\xi'(k') - \frac{1}{2}k'^2\right) - 2\left(\xi'(k') - \frac{1}{2}k'^2\right); \tag{3.6}$$

$$\sigma' = -\frac{1}{k'}\,\xi'\left(\xi'(k') - \frac{1}{2}k'^2\right) - (n+1)\left(\xi'(k') - \frac{1}{2}k'^2\right). \tag{3.7}$$

Proof. By using (3.4), we find immediately (3.5) and (3.6). Taking a trace in (3.5), we have

$$\rho'(Y, \xi') = -\left[\frac{1}{k'}\,\xi'\left(\xi'(k') - \frac{1}{2}k'^2\right) + (n+1)\left(\xi'(k') - \frac{1}{2}k'^2\right)\right]\eta'(Y), \tag{3.8}$$

which implies (3.7). □

The equality (3.8) shows that every unit vector in $D^\perp(p)$ is an eigenvector of the Ricci operator ρ' with one and the same eigenvalue $\sigma'(p)$.

If x_0 is a unit vector in $D(p)$, then the Riemannian sectional curvature of $\operatorname{span}\{x_0, \xi'\}$ may only depend on the point $p \in M$:

$$-R'(x_0, \xi', \xi', x_0) = \frac{\sigma' - \varkappa'}{2(n-1)}. \tag{3.9}$$

The first step in the study of Kähler–Lorentz manifolds with B_0-distributions is to describe the flat case.

Let (M, h', J, D) ($\dim M \geq 6$) be a flat Kähler–Lorentz manifold with B_0-distribution $D(D^\perp = \mathrm{span}\{\xi', J\xi'\})$. Then Lemma 3.3 implies that

$$\xi'(k') = \frac{1}{2}k'^2. \tag{3.10}$$

Taking into account Lemma 3.2 it follows that

$$p^{*\prime} = -\frac{1}{2}k'. \tag{3.11}$$

Then (3.4) in view of (3.10) and (3.11) implies that

$$\nabla'_x\xi' = -\frac{1}{2}k'\,x, \qquad h'(x, \xi') = 0. \tag{3.12}$$

Hence the integral submanifolds $Q_1^{2(n-1)}$ of the distribution Δ, perpendicular to ξ', are totally umbilic submanifolds of M with time-like normals ξ'.

Let $(\mathbb{C}^n = \{\mathbf{Z} = (z^1, \dots, z^{n-1}; z^n)\}, J)$ be the standard n-dimensional complex vector space with complex structure J and h' be the Kähler metric of signature $(2(n-1), 2)$, defined by

$$h'(\mathbf{Z}, \mathbf{Z}) = |z^1|^2 + \cdots + |z^{n-1}|^2 - |z^n|^2.$$

We call h' the *canonical flat Kähler–Lorentz metric* and $(\mathbb{C}^n, h', J) = (\mathbb{R}_2^{2(n-1)}, h', J)$ the *canonical flat Kähler–Lorentz manifold*.

Next we describe the B_0-distributions in $(\mathbb{C}^n, h', J)$.

Let D ($D^\perp = \mathrm{span}\{\xi', J\xi'\}$) be a B_0-distribution in $(\mathbb{C}^n, h', J)$. According to Definition 3.1 ξ' is a time-like geodesic vector field with respect to the flat Levi-Civita connection ∇' of h'. Then the integral curves of ξ' are straight lines. Since h' is flat, then the integral submanifolds $Q_1^{2(n-1)}$ of the distribution Δ, perpendicular to ξ', are totally umbilical with time-like normals ξ'. Applying the standard theorem for totally umbilical submanifolds (with time-like normals) of the manifold $(\mathbb{C}^n, h', J)$, we obtain that $Q_1^{2(n-1)}$ is locally a part of a hypersphere $H_1^{2(n-1)}(\mathbf{Z}_0, r) : h'(\mathbf{Z} - \mathbf{Z}_0, \mathbf{Z} - \mathbf{Z}_0) = -r^2$, $r > 0$. All these hyperspheres are orthogonal to the integral curves of ξ', i.e. $Q_1^{2(n-1)}$ are the concentric hyperspheres

$$H_1^{2(n-1)}(\mathbf{Z}_0, r) : h'(\mathbf{Z} - \mathbf{Z}_0, \mathbf{Z} - \mathbf{Z}_0) = -r^2, \quad \mathbf{Z}_0 = \mathrm{const}.$$

Choosing $\mathbf{Z}_0$ at the origin O of $\mathbb{C}^n$, we obtain

Canonical example of a flat Kähler–Lorentz manifold with B_0-distribution:

$$(\mathbb{T}_1^{n-1}, h', J, D),$$

where $\mathbb{T}_1^{n-1}$ is the time-like domain in $\mathbb{C}^n$

$$\mathbb{T}_1^{n-1} = \{\mathbf{Z} \in \mathbb{C}^n \mid h'(\mathbf{Z}, \mathbf{Z}) < 0\}$$

and

$$\xi' = \frac{\mathbf{Z}}{\sqrt{-h'(\mathbf{Z}, \mathbf{Z})}}, \quad \mathbf{Z} \in \mathbb{T}_1^{n-1}.$$

Now let (M, h', J, D) ($\dim M = 2n \geq 6$) be a flat Kähler–Lorentz manifold with B_0-distribution D ($D^\perp = \mathrm{span}\{\xi', J\xi'\}$). Since the Levi-Civita connection ∇' of h' is flat and $\nabla' J = 0$, then there exists a local holomorphic isometry ϕ of (M, h', J) onto $(\mathbb{C}^n, h', J)$. Since ϕ transforms the B_0-distribution D into a B_0-distribution, then we have

Proposition 3.4. *Any flat Kähler–Lorentz manifold with B_0-distribution is locally equivalent to the canonical example* $(\mathbb{T}_1^{n-1}, h', J, D)$.

In order to make computations in local holomorphic coordinates we need some formulas concerning the structures on $\mathbb{T}_1^{n-1}$.

Let in $\mathbb{C}^n = \{\mathbf{Z} = (z^1, \dots, z^n)\}$ $(n \geqq 2)$ $\partial_\alpha := \frac{\partial}{\partial z^\alpha}$, $\partial_{\bar\alpha} := \frac{\partial}{\partial z^{\bar\alpha}} = \overline{\frac{\partial}{\partial z^\alpha}}$, $\alpha = 1, \dots, n$. Further, the indices $\alpha, \beta, \dots$ will run over $1, \dots, n$.

The canonical flat Kähler–Lorentz metric h' has the following local components:

$$h'_{\alpha\bar\beta} = \begin{cases} \dfrac{1}{2} & \alpha = \beta = 1, \dots, n-1; \\ -\dfrac{1}{2} & \alpha = \beta = n; \\ 0 & \alpha \neq \beta. \end{cases}$$

Then

$$h'(\mathbf{Z}, \mathbf{Z}) = |z^1|^2 + \cdots + |z^{n-1}|^2 - |z^n|^2 = 2h'_{\alpha\bar\beta}\, z^\alpha z^{\bar\beta},$$

where the summation convention is assumed.

The distance function $-r^2 = h'(\mathbf{Z}, \mathbf{Z})$ in the domain $\mathbb{T}_1^{n-1}$ is given by

$$-r^2 = 2h'_{\alpha\bar\beta} z^\alpha z^{\bar\beta} < 0, \quad r > 0. \tag{3.13}$$

The vector field $\xi' = \frac{1}{r}\mathbf{Z}$ at the point $p \in \mathbb{T}_1^{n-1}$ with position vector $\mathbf{Z}$ has local components

$$\eta'^\alpha = \frac{1}{r}\, \delta^\alpha_\sigma z^\sigma,$$

where the δ^σ_α are Kronecker deltas.

Taking into account (3.13), we find the local components of the corresponding 1-form η':

$$\eta'_\alpha = \eta'^{\bar\sigma} h'_{\alpha\bar\sigma} = \frac{1}{r}\, h'_{\alpha\bar\beta} z^{\bar\beta} = -r_\alpha. \tag{3.14}$$

Hence

$$\begin{aligned} &\eta' = -\mathrm{d}r, \qquad \xi' = \frac{\mathrm{d}}{\mathrm{d}r}; \\ &\eta'(\xi') = h'(\xi', \xi') = \frac{1}{r^2}\, h'(\mathbf{Z}, \mathbf{Z}) = -1. \end{aligned} \tag{3.15}$$

By differentiating (3.13) we obtain

$$h'_{\alpha\bar\beta} = \frac{1}{2}\, \partial_\alpha \partial_{\bar\beta}(-r^2).$$

On the other hand, differentiating (3.14), we have

$$\partial_{\bar\beta}\eta'_\alpha = \nabla'_{\bar\beta}\eta'_\alpha = \frac{1}{r}\, (h'_{\alpha\bar\beta} + \eta'_\alpha \eta'_{\bar\beta}).$$

Proposition 3.5. *Let $f(t)$, $t < 0$ be a real $\mathcal{C}^\infty$-function satisfying the inequalities:*

$$f'(t) > 0, \quad f'(t) + t f''(t) < 0.$$

Then

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} f(-r^2)$$

are the local components of a Kähler metric g.

Proof. By using (3.13) and (3.15), we calculate

$$\partial_{\bar\beta} f(-r^2) = 2f' h'_{\alpha\bar\beta} z^\alpha.$$

Differentiating the last equality, we find

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} f(-r^2) = 2f' h'_{\alpha\bar\beta} + 4r^2 f'' \eta'_\alpha \eta'_{\bar\beta}.$$

Hence,

$$g = 2f'h' + 2r^2 f''(\eta' \otimes \eta' + \tilde\eta' \otimes \tilde\eta'). \tag{3.16}$$

Now, let $p \in \mathbb{T}_1^{n-1}$ and $T_p(\mathbb{T}_1^{n-1}) = (D(p) \oplus D^\perp(p))$. The equality (3.16) implies that

$$g(x_0, x_0) = 2f'h'(x_0, x_0), \quad x_0 \in D(p); \tag{3.17}$$

$$g(\xi', \xi') = g(J\xi', J\xi') = -2(f' + (-r^2) f''). \tag{3.18}$$

The first condition of the proposition and (3.17) imply that the restriction of g onto D is positive definite. The second condition of the proposition and (3.18) give that the restriction of g onto $D^\perp$ is also positive definite. Hence g is a positive definite metric. Since $g = \partial\bar\partial f(-r^2)$, then g is a Kähler metric. □

4. Kähler manifolds of quasi-constant holomorphic sectional curvatures with $a + k^2 < 0$

In this section we prove the main theorem, which clarifies the connection between the Kähler metrics introduced in Section 3 and a class of Kähler manifolds of quasi-constant holomorphic sectional curvatures.

Let (M, g, J, D) ($\dim M = 2n \geq 6$) be a Kähler manifold with B_0-distribution D ($D^\perp = \operatorname{span}\{\xi, J\xi\}$) with functions k, p^*, given by (2.2).

If u, v are proper $\mathcal{C}^\infty$-functions of the distribution Δ (cf. [3]), i.e. $\mathrm{d}u = \xi(u)\,\eta$, $\mathrm{d}v = \xi(v)\,\eta$, we consider the metric

$$h' = \mathrm{e}^{2u}\left(g - (\mathrm{e}^{2v}+1)(\eta\otimes\eta + \tilde\eta \otimes \tilde\eta)\right), \tag{4.1}$$

which is positive definite on D and negative definite on $D^\perp$.

Lemma 4.1. *Let* (M, g, J, D) ($\dim M = 2n \geq 6$) *be a Kähler manifold with* B_0*-distribution* D ($D^\perp = \operatorname{span}\{\xi, J\xi\}$). *Then the metric* h'*, given by* (4.1)*, is Kähler–Lorentz if and only if*

$$\xi(u) = -\frac{k(\mathrm{e}^{2v}+1)}{2}. \tag{4.2}$$

Proof. From (4.1) we find the Kähler form Θ of the metric h':

$$\Theta = \mathrm{e}^{2u}\left(\Omega - (\mathrm{e}^{2v}+1)\eta \wedge \tilde\eta\right).$$

The last equality, (2.1) and (2.2), imply that

$$\mathrm{d}\Theta = \mathrm{e}^{2u}\left(2\xi(u) + k(\mathrm{e}^{2v}+1)\right)\eta \wedge \Omega,$$

which implies the assertion of the lemma. □

We set

$$\xi' = \mathrm{e}^{-(u+v)}\,\xi, \qquad \eta' = -\mathrm{e}^{u+v}\,\eta. \tag{4.3}$$

Then η' is the 1-form corresponding to ξ' with respect to h' and $\eta'(\xi') = -1$.

Lemma 4.2. *Let* (M, g, J, D) ($\dim M = 2n \geq 6$) *be a Kähler manifold with* B_0*-distribution* D ($D^\perp = \operatorname{span}\{\xi, J\xi\}$). *If*

$$h' = \mathrm{e}^{2u}\left(g - (\mathrm{e}^{2v}+1)(\eta\otimes\eta + \tilde\eta \otimes \tilde\eta)\right),$$

where

$$dv = \xi(v)\,\eta, \qquad du = -\frac{k(e^{2v}+1)}{2}\,\eta$$

and

$$\xi' = e^{-(u+v)}\,\xi, \qquad \eta' = -e^{u+v}\,\eta,$$

then (M, h', J, D) $(D^{\perp} = \mathrm{span}\{\xi', J\xi'\})$ *is a Kähler–Lorentz manifold with space-like* B_0*-distribution* D.

Proof. Let ∇', ∇ be the Levi-Civita connections of the metrics h', g, respectively. Then

$$\begin{aligned}\nabla'_X Y = \nabla_X Y &+ \xi(u)\{\eta(X)Y + \eta(Y)X + \tilde{\eta}(X)JY + \tilde{\eta}(Y)JX\} + \xi(v-u)\{[\eta(X)\eta(Y) - \tilde{\eta}(X)\tilde{\eta}(Y)]\xi \\ &+ [\eta(X)\tilde{\eta}(Y) + \tilde{\eta}(X)\eta(Y)]J\xi\}, \quad X, Y \in \mathfrak{X}M. \end{aligned} \tag{4.4}$$

From (4.4) it follows that

$$\nabla'_X \xi' = e^{-(u+v)}\left(\xi(u) + \frac{k}{2}\right)\left[X - \eta(X)\xi - \tilde{\eta}(X)J\xi\right] + e^{-(u+v)}\left(\xi(u+v) - p^*\right)\tilde{\eta}(X)J\xi, \quad X \in \mathfrak{X}M.$$

The above equality can be written in the form

$$\nabla'_X \xi' = -\frac{k'}{2}[X + \eta'(X)\xi' + \tilde{\eta}'(X)J\xi'] - p^{*\prime}\tilde{\eta}'(X)J\xi', \quad X \in \mathfrak{X}M, \tag{4.5}$$

where

$$\begin{aligned} k' &= -2e^{-(u+v)}\left(\xi(u) + \frac{k}{2}\right), \\ p^{*\prime} &= e^{-(u+v)}(\xi(u+v) - p^*), \end{aligned} \tag{4.6}$$

i.e. D is a space-like B_0-distribution with functions k' and $p^{*\prime}$. □

Because of (4.2)

$$\xi(u) + \frac{k}{2} = -\frac{1}{2}e^{2v}k.$$

Then (4.6) gives the following relation between k' and k:

$$k' = e^{v-u}\,k. \tag{4.7}$$

Let the tensors π', Φ'_1, Φ'_2, $\Phi' = \Phi'_1 + \Phi'_2$ and Ψ' of type (1,3) with respect to the structure (h', ξ', η') be determined as in (2.3)–(2.5). If g and h' are related as in Lemma 4.2, then

$$\begin{aligned} \pi' + 2\Phi' + \Psi' &= e^{2u}(\pi - 2\Phi + \Psi), \\ \Phi'_1 + \frac{1}{2}\Psi' &= -e^{2u}\left(\Phi_1 - \frac{1}{2}\Psi\right), \\ \Phi'_2 + \frac{1}{2}\Psi' &= e^{2(u+v)}\left(\Phi_2 - \frac{1}{2}\Psi\right), \\ \Psi' &= -e^{2(u+v)}\,\Psi. \end{aligned} \tag{4.8}$$

Proposition 4.3. *Let* (M, g, J, D) $(\dim M = 2n \geq 6)$ *be a Kähler manifold of quasi-constant holomorphic sectional curvatures with* B_0*-distribution* D $(D^{\perp} = \mathrm{span}\{\xi, J\xi\})$ *and*

$$a + k^2 < 0.$$

If the structure (h', ξ', η') *is determined as in Lemma* 4.2 *by the proper function*

$$e^{2v} = -\frac{a+k^2}{k^2},$$

then h' *is a flat Kähler–Lorentz metric.*

Proof. By direct computations from (4.4) in view of (2.3), (2.4) and (2.5) we find

$$\begin{aligned} R' - R \; = \; & -2k\xi(u)(\pi - 2\Phi + \Psi) - 4k\xi(v)\left(\Phi_1 - \frac{1}{2}\Psi\right) - 4(\xi^2(u) - p^*\xi(u))\left(\Phi_2 - \frac{1}{2}\Psi\right) \\ & - (\xi^2(u+v) - p^*\xi(u+v))\,\Psi. \end{aligned} \tag{4.9}$$

Taking into account that $R = a\pi + b\Phi + c\Psi$ and (4.8), we obtain from (4.9) the curvature tensor R' of h' in the form

$$R' = A(\pi' + 2\Phi' + \Psi') + B_1\left(\Phi_1' + \frac{1}{2}\Psi'\right) + B_2\left(\Phi_2' + \frac{1}{2}\Psi'\right) + C\Psi', \tag{4.10}$$

where

$$\begin{aligned} & e^{2u}A = a - 2k\xi(u), \qquad e^{2(u+v)}C = -(a+b+c) + \xi^2(u+v) - p^*\xi(u+v), \\ & e^{2u}B_1 = -(2a+b) + 4k\xi(v), \qquad e^{2(u+v)}B_2 = 2a + b - 4(\xi^2(u) - p^*\xi(u)). \end{aligned} \tag{4.11}$$

Taking into account (4.7), (4.10) and (4.2), we find

$$e^{2u}(A - k'^2) = a + k^2. \tag{4.12}$$

Then (4.12) and (4.7) imply

$$e^{2u}A = e^{2v}k^2 + a + k^2.$$

Under the conditions of the proposition we obtain $A = 0$ and $\xi(u) = \frac{a}{2k}$.

Differentiating the equality $e^{2v} = -\frac{a+k^2}{k^2}$, because of (2.6), we obtain

$$\xi(k) + \frac{1}{2}k^2 + k\xi(v) = 0. \tag{4.13}$$

On the other hand, $\xi' = e^{-(u+v)}\,\xi$ and (4.7) imply

$$\xi(k) + \frac{1}{2}k^2 + k\xi(v) = e^{2u}\left(\xi'(k') - \frac{1}{2}k'^2\right).$$

Thus, from the equality (4.13), we get

$$\xi'(k') = \frac{1}{2}\,k'^2.$$

Now from (3.6) and (3.7) it follows that $\varkappa' = \sigma' = 0$.

Replacing into (4.10) the quadruples ξ, x_0, x_0, ξ and x_0, ξ, ξ, x_0, where $h'(x_0, x_0) = 1$, in view of (3.9), we obtain

$$0 = \frac{\sigma' - \varkappa'}{2(n-1)} = \frac{1}{8}B_1 = \frac{1}{8}B_2.$$

Replacing into (4.10) the quadruple $\xi, J\xi, J\xi, \xi$, we get

$$0 = \varkappa' = C,$$

i.e. $R' = 0$. □

Let now (M, h', J, D) $(\dim M = 2n \geq 6)$ be a Kähler–Lorentz manifold with space-like B_0-distribution D $(D^\perp = \mathrm{span}\{\xi', J\xi'\})$ with functions k' and $p^{*\prime}$, determined in Lemma 3.2.

If u, v are proper $\mathcal{C}^\infty$-functions of the distribution Δ, i.e. $du = -\xi'(u)\eta'$, $dv = -\xi'(v)\eta'$, we consider the metric

$$g = e^{-2u}(h' + (e^{-2v} + 1)(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')). \tag{4.14}$$

Taking into account (3.4), analogously to Lemma 4.1, we have

Lemma 4.4. *Let* (M, h', J, D) $(\dim M = 2n \geq 6)$ *be a Kähler–Lorentz manifold with space-like* B_0*-distribution* D $(D^{\perp} = \operatorname{span}\{\xi', J\xi'\})$. *Then the metric* g*, given by* (4.14)*, is Kähler if and only if*

$$\xi'(u) = -\frac{k'(e^{-2v}+1)}{2}.$$

Further, we set $\xi = e^{u+v}\xi'$, $\eta = -e^{-(u+v)}\eta'$. Analogously to (4.7) we have

$$k = e^{u-v}k'. \tag{4.15}$$

Lemma 4.5. *Let* (M, h', J, D) $(\dim M = 2n \geq 6)$ *be a Kähler–Lorentz manifold with space-like* B_0*-distribution* D $(D^{\perp} = \operatorname{span}\{\xi', J\xi'\})$. *If*

$$g = e^{-2u}\left(h' + (e^{-2v}+1)(\eta'\otimes\eta' + \tilde{\eta}'\otimes\tilde{\eta}')\right),$$

where

$$dv = -\xi'(v)\,\eta', \qquad du = \frac{k'(e^{-2v}+1)}{2}\,\eta'$$

and

$$\xi = e^{u+v}\,\xi', \qquad \eta = -e^{-(u+v)}\,\eta',$$

then (M, g, J, D) $(D^{\perp} = \operatorname{span}\{\xi, J\xi\})$ *is a Kähler manifold with* B_0*-distribution* D.

Proposition 4.6. *Let* $(\mathbb{T}_1^{n-1}, h', J, D)$ $(\dim \mathbb{T}_1^{n-1} = 2n \geq 6)$ *be the canonical example of a flat Kähler–Lorentz manifold. If the structure* (g, ξ, η) *is determined as in Lemma 4.5, then* g *is a Kähler metric of quasi-constant holomorphic sectional curvatures and* $a + k^2 < 0$.

Proof. Taking into account (4.14) we find the relation (4.4) between the Levi-Civita connections ∇' and ∇ of h' and g, respectively. Then the corresponding relation between the curvature tensors R' and R is given by (4.9). Since $R' = 0$, then (4.9) gives the tensor R in the form

$$R = A^*(\pi - 2\Phi + \Psi) + B_1^*\left(\Phi_1 - \frac{1}{2}\Psi\right) + B_2^*\left(\Phi_2 - \frac{1}{2}\Psi\right) + C^*\Psi.$$

Replacing the quadruples ξ, x_0, x_0, ξ; x_0, ξ, ξ, x_0, where $g(x_0, x_0) = 1$, in the last equality, we get

$$\frac{1}{8}B_1^* = R(\xi, x_0, x_0, \xi) = R(x_0, \xi, \xi, x_0) = \frac{1}{8}B_2^*.$$

Hence the curvature tensor R has the form $R = a\pi + b\Phi + c\Psi$, i.e. the metric g is of quasi-constant holomorphic sectional curvatures.

To prove $a + k^2 < 0$, we consider (4.9). Since $a = 2k\xi(u)$, $\xi = e^{u+v}\xi'$, in view of (4.15) and Lemma 4.4, we find

$$a = -e^{2u}k'^2(e^{-2v}+1) = -k^2 - e^{2u}k'^2.$$

Hence $a + k^2 < 0$. $\square$

Theorem 4.7. *Any Kähler metric* $g = \partial\bar{\partial}f(-r^2)$, $-r^2$ *being the time-like distance function in* $\mathbb{T}_1^{n-1}$ $(n \geq 3)$*, is of quasi-constant holomorphic sectional curvatures and function* $a + k^2 < 0$.

Conversely, every Kähler manifold (M, g, J, D) $(\dim M = 2n \geq 6)$ *of quasi-constant holomorphic sectional curvatures with* B_0*-distribution satisfying the condition* $a + k^2 < 0$ *is locally equivalent to* $(\mathbb{T}_1^{n-1}, g, J, D)$ *with the canonical* B_0*-distribution and* $g = \partial\bar{\partial}f(-r^2)$.

Proof. Let the Kähler metric g be given as in (3.16). Putting

$$e^{-2u} = 2f', \qquad e^{-2v} + 1 = \frac{r^2 f''}{f'},$$

we calculate

$$\xi'(u) = \frac{\mathrm{d}u}{\mathrm{d}r} = \frac{rf''}{f'} = \frac{1}{r}(\mathrm{e}^{-2v} + 1) = -\frac{k'(\mathrm{e}^{-2v} + 1)}{2}.$$

Then we can apply Proposition 4.6 and conclude that the structure (g, J, ξ, η) is of quasi-constant holomorphic sectional curvatures with B_0-distribution and function $a + k^2 < 0$.

For the inverse, let (M, g, J, D) $(\dim M = 2n \geq 6)$ be a Kähler manifold with B_0-distribution and function $a + k^2 < 0$. We construct the metric h' as in Lemma 4.2 by the proper function $\mathrm{e}^{2v} = -\frac{a+k^2}{k^2}$. Applying Proposition 4.6 we obtain that the Kähler metric h' is flat and the given manifold is locally equivalent to the canonical flat Kähler–Lorentz manifold $(\mathbb{T}_1^{n-1}, h', J, D)$.

Further we write the equality (4.1) in the form

$$g = \mathrm{e}^{-2u}\left(h' + (\mathrm{e}^{-2v} + 1)(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')\right) \tag{4.16}$$

and put

$$f(-r^2) = \frac{1}{2}\int \mathrm{e}^{-2u}\mathrm{d}(-r^2). \tag{4.17}$$

From (4.17) we have $\mathrm{e}^{-2u} = 2f'$. Using Lemma 4.5 we find $\xi'(u) = -\frac{k'(\mathrm{e}^{-2v}+1)}{2}$ and $\xi' = \frac{\mathrm{d}}{\mathrm{d}r}$, $k' = -\frac{2}{r}$. Then $\mathrm{e}^{-2v} + 1 = \frac{r^2 f''}{f'}$ and (4.16) becomes

$$g = 2f'\left(h' + \frac{r^2 f''}{f'}(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')\right).$$

Hence $g = \partial\bar{\partial} f(-r^2)$ with potential function (4.17). □

As an application of Theorem 4.7 we shall find the Kähler metrics of constant holomorphic sectional curvatures, defined in the manifold $(\mathbb{T}_1^{n-1}, h', J, D)$ by the condition $g = \partial\bar{\partial} f(-r^2)$.

Let g be a metric given by (4.16). Then (4.9) gives the relation between the curvature tensor R of g and the tensor $R' = 0$ of h' in $\mathbb{T}_1^{n-1}$. Since the coefficients A, B_1, B_2, C in (4.10) are all zero, then (4.11) implies

$$\begin{aligned} &a = 2k\xi(u);\\ &2a + b = 4k\xi(v) = 4[\xi^2(u) - p^*\xi(u)];\\ &a + b + c = \xi^2(u + v) - p^*\xi(u + v). \end{aligned} \tag{4.18}$$

Since D is a B_0-distribution, then g is a Kähler metric of constant holomorphic sectional curvatures if and only if $b = 0$. Because of $\xi = \mathrm{e}^{u+v}\xi' = \mathrm{e}^{u+v}\frac{\mathrm{d}}{\mathrm{d}r}$ and (4.18), the condition $b = 0$ is equivalent to the relation

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\mathrm{d}v}{\mathrm{d}r}. \tag{4.19}$$

Further, taking into account (4.18) and (2.2), we obtain successively

$$k\xi(u) = \xi^2(u) - p^*\xi(u) = \xi^2(u) + \frac{\xi(k) + k^2}{k^2}\xi(u),$$

which in view of (4.19) and the relation $k = \mathrm{e}^{u-v}k' = -\frac{2\mathrm{e}^{u-v}}{r}$ implies that

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + 2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 - \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = 0. \tag{4.20}$$

Solving (4.20), we find

$$\mathrm{e}^{2u} = \mathrm{e}^{2u_0}|r^2 + a_0|, \quad a_0 = \text{const}, u_0 = \text{const}. \tag{4.21}$$

Since $a + k^2 < 0$, then $a < 0$ and the equality $a = 2k\xi(u) = -\frac{4}{r}e^{2u}\frac{du}{dr} = -\frac{4e^{2u}}{r^2+a_0}$ implies that $r^2 + a_0 > 0$.
On the other hand, using the relation (4.2), we find $e^{-2v} = -\frac{a_0}{r^2+a_0} > 0$ and $a_0 < 0$. Putting $a_0 = -r_0^2$, we have

$$e^{-2v} = \frac{r_0^2}{r^2 - r_0^2}. \tag{4.22}$$

Finally, the equality $a = 2k\xi(u)$ gives that $e^{-2u_0} = -\frac{4}{a}$.
Now, from (4.21) and (4.22) we obtain
Examples of Kähler space forms with B_0-distribution and $a + k^2 < 0$:
All Kähler metrics g of constant holomorphic sectional curvatures $a < 0$, given in $\mathbb{T}_1^{n-1}$ by (4.16), are

$$g = -\frac{4}{a(r^2 - r_0^2)}\left(h' + \frac{r^2}{r^2 - r_0^2}(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')\right), \quad r_0 = \text{const} > 0, r > r_0. \tag{4.23}$$

The potential function of the above metrics up to a constant is

$$f(-r^2) = \frac{2}{a}\ln(r^2 - r_0^2), \quad r_0 = \text{const} > 0, r > r_0.$$

Hence

$$g = \frac{2}{a}\,\partial\bar{\partial}\,\ln(r^2 - r_0^2), \quad r_0 > 0, r > r_0.$$

One of these metrics is most remarkable:

$$g = \frac{4}{r^2 - 1}\left(h' + \frac{r^2}{r^2 - 1}(\eta' \otimes \eta' + \tilde{\eta}' \otimes \tilde{\eta}')\right), \quad r > 1. \tag{4.24}$$

This metric is defined in the hyperbolic unit "disc" $\mathbb{D}_1^{n-1}(1) : h'(\mathbf{Z}, \mathbf{Z}) < -1$ and is of constant holomorphic sectional curvatures $a = -1$.

5. The geometric meaning of the function $a + k^2$ in Kähler manifolds of quasi-constant holomorphic sectional curvatures

Let (M, g, J, D) ($\dim M = 2n \geq 6$) be a Kähler manifold of quasi-constant holomorphic sectional curvatures with B_0-distribution $D(p)$ ($D^\perp(p) = \text{span}\{\xi, J\xi\}$), $p \in M$.
In this section we study the geometric structure of the integral submanifolds of the distribution

$$\Delta(p) = \{X \in T_pM \mid \eta(X) = 0\}, \quad p \in M.$$

Because of (2.1), we have

$$d\eta = 0; \qquad d\tilde{\eta} = k\Omega + \frac{1}{k}\eta \wedge \tilde{\eta}. \tag{5.1}$$

Let Q^{2n-1} be an arbitrary integral submanifold of the distribution Δ and ξ be the unit vector field, normal to Q^{2n-1}. Applying the Weingarten and Gauss equations to the submanifolds Q^{2n-1}, we have

$$\nabla_x \xi = \frac{k}{2}x + \frac{1}{k}\left(\xi(k) + \frac{k^2}{2}\right)\tilde{\eta}(x)J\xi, \quad x \in \mathfrak{X}\Delta; \tag{5.2}$$

$$\nabla_x y = \mathcal{D}_x y + h(x, y)\xi, \quad x, y \in \mathfrak{X}\Delta, \tag{5.3}$$

where $\mathcal{D}$ is the induced Levi-Civita connection and h is the second fundamental tensor on Q^{2n-1}.
According to (2.2), $k = \text{const}$ on Q^{2n-1}. From (5.2) it follows that

$$h = -\frac{k}{2}\,g - \frac{1}{k}\left(\xi(k) + \frac{k^2}{2}\right)\tilde{\eta} \otimes \tilde{\eta}.$$

The standard almost contact Riemannian structures $(g, \varphi, \tilde{\xi}, \tilde{\eta})$ induced on the manifold Q^{2n-1} are [8,9]:

$$\begin{aligned} &\tilde{\xi} := J\xi; \qquad \tilde{\eta} = g(x, \tilde{\xi}),\\ &\varphi x := Jx + \tilde{\eta}(x)\xi, \quad x \in \mathfrak{X}\Delta. \end{aligned} \tag{5.4}$$

Taking into account (5.2), in view of (2.7), we find

$$\mathcal{D}_x \tilde{\xi} = \frac{k}{2}\,\varphi x, \quad x \in \mathfrak{X}\Delta, \tag{5.5}$$

$$(\mathcal{D}_x \varphi)(y) = \frac{k}{2}\left(\tilde{\eta}(y)x - g(x, y)\tilde{\xi}\right), \quad x, y \in \mathfrak{X}\Delta. \tag{5.6}$$

According to (5.5) and (5.6), any integral submanifold Q^{2n-1} of the distribution Δ is an α-Sasakian manifold with $\alpha = \frac{k}{2}$.

More precisely we have

Proposition 5.1. *Let* (M, g, J, D) $(\dim M = 2n \geq 6)$ *be a Kähler manifold of quasi-constant holomorphic sectional curvatures with* B_0*-distribution* D $(D^{\perp} = \operatorname{span}\{\xi, J\xi\})$.

Then any integral submanifold Q^{2n-1} *of the distribution* Δ *is a* $\frac{k}{2}$*-Sasakian space form of type* $\begin{cases} \text{I}, \\ \text{II}, \\ \text{III}, \end{cases}$ *if and only if* $\begin{cases} a + k^2 > 0, \\ a + k^2 = 0, \\ a + k^2 < 0, \end{cases}$ *respectively.*

Proof. From (5.3), (5.2), (5.4) and (5.6) we find the relation between the curvature tensors R and K of M^{2n} and Q^{2n-1}, respectively:

$$\begin{aligned} R(x, y, z, u) \;=\; & K(x, y, z, u) - \frac{1}{4}k^2[g(y, z)g(x, u) - g(x, z)g(y, u)] \\ & - \frac{1}{2}\left(\xi(k) + \frac{1}{2}k^2\right)[g(y, z)\tilde{\eta}(x)\tilde{\eta}(u) + g(x, u)\tilde{\eta}(y)\tilde{\eta}(z) \\ & - g(x, z)\tilde{\eta}(y)\tilde{\eta}(u) - g(y, u)\tilde{\eta}(x)\tilde{\eta}(z)], \quad x, y, z, u \in \mathfrak{X}\Delta. \end{aligned} \tag{5.7}$$

Since (M, g, J, D) is of quasi-constant holomorphic sectional curvatures, then

$$R = a\pi + b\Phi + c\Psi. \tag{5.8}$$

Taking into account (5.8), the equality (5.7) becomes

$$\begin{aligned} K(x, y, z, u) \;=\; & \frac{a + k^2}{4}[g(y, z)g(x, u) - g(x, z)g(y, u)] \\ & + \frac{a}{4}[g(\varphi y, z)g(\varphi x, u) - g(\varphi x, z)g(\varphi y, u) - 2g(\varphi x, y)g(\varphi z, u) \\ & - g(y, z)\tilde{\eta}(x)\tilde{\eta}(u) - g(x, u)\tilde{\eta}(y)\tilde{\eta}(z) \\ & + g(x, z)\tilde{\eta}(y)\tilde{\eta}(u) + g(y, u)\tilde{\eta}(x)\tilde{\eta}(z)], \quad x, y, z, u \in \mathfrak{X}\Delta. \end{aligned} \tag{5.9}$$

Comparing (5.9) with the equality from Proposition 2.1 we obtain

$$c + 3\alpha^2 = a + k^2, \qquad c - \alpha^2 = a. \tag{5.10}$$

Hence any integral submanifold Q^{2n-1} of Δ is an α-Sasakian space form with

$$\alpha = \frac{k}{2}, \qquad c = a + \frac{k^2}{4}.$$

Now the relation (5.10) gives the assertion. □

The above statement allows us to obtain examples of α-Sasakian (Sasakian) manifolds of constant φ-holomorphic sectional curvatures c satisfying the condition $c + 3\alpha^2 < 0$ $(c + 3 < 0)$ as hypersurfaces of the Kähler space form $(\mathbb{T}_1^{n-1}, g, J, D)$, with g given by (4.24).

Let $(\mathbb{T}_1^{n-1}, h', J, D)$ be the canonical example of a flat Kähler–Lorentz manifold with B_0-distribution D and g be the Kähler metric of constant holomorphic sectional curvatures -1, given by (4.24). We denote by $H_1^{2(n-1)}(O, r)$ any hypersphere in $\mathbb{T}_1^{n-1}$, centered at the origin O and with radius $r > 1$, given by

$$H_1^{2(n-1)}(O, r) = \{\mathbf{Z} \in \mathbb{T}_1^{n-1} \mid h'(\mathbf{Z}, \mathbf{Z}) = -r^2\}.$$

Then an easy verification shows that $H_1^{2(n-1)}(O, r)$ with the induced from $(\mathbb{T}_1^{n-1}, g, J, D)$ structure $(g, \varphi, \tilde{\xi}, \tilde{\eta})$ is an α-Sasakian manifold with constant φ-holomorphic sectional curvatures c such that

$$\alpha = \frac{1}{2r}, \qquad c + 3\alpha^2 = -\frac{r^2 - 1}{r^2}.$$

Further we give a direct construction of examples of Sasakian structures with prescribed φ-holomorphic sectional curvatures c of type $c + 3 < 0$ using as a base the hypersphere $H_1^{2(n-1)}(O, r = 1) = H_1^{2(n-1)}(1)$.

Let $(h', \varphi, \tilde{\xi}, \tilde{\eta})$ be the induced from $(\mathbb{T}_1^{n-1}, h', J, D)$ onto $H_1^{2(n-1)}(1)(-1)$-Sasakian structure with $h'(\tilde{\xi}, \tilde{\xi}) = -1$. We introduce the following family of Riemannian metrics

$$g = q^2(h' + (1 + q^2)\tilde{\eta} \otimes \tilde{\eta}), \quad q = \text{const} > 0 \tag{5.11}$$

on $H_1^{2(n-1)}(1)$. Any of these metrics generates the corresponding unit vector field $\bar{\xi}$ and 1-form $\bar{\eta}$ determined by

$$\bar{\xi} = \frac{1}{q^2}\tilde{\xi}, \qquad \bar{\eta} = -q^2\tilde{\eta}.$$

In a straightforward way we obtain that $(H_1^{2(n-1)}(1), g, \varphi, \bar{\xi}, \bar{\eta})$ is a Sasakian manifold. Further, by direct computations we find that the Sasakian structure $(g, \varphi, \bar{\xi}, \bar{\eta})$ is of constant φ-holomorphic sectional curvatures c satisfying the relation

$$c + 3 = -\frac{4}{q^2}.$$

Thus we obtained

Examples of Sasakian space forms with prescribed φ-holomorphic sectional curvatures c satisfying the condition $c + 3 < 0$:

$$(H_1^{2(n-1)}(1), g, \varphi, \bar{\xi}, \bar{\eta}): \quad \bar{\xi} = -\frac{c+3}{4}\,\tilde{\xi}, \quad \bar{\eta} = \frac{4}{c+3}\,\tilde{\eta}, \quad g = -\frac{4}{c+3}\left(h' + \frac{c-1}{c+3}\,\tilde{\eta} \otimes \tilde{\eta}\right).$$

6. Kähler structures on rotational hypersurfaces

In this section we consider three types of rotational hypersurfaces in spaces with definite or indefinite flat metrics, which will be endowed with Kähler structures of quasi-constant holomorphic sectional curvatures.

In Sections $\begin{cases} 6.1, \\ 6.2, \\ 6.3 \end{cases}$ we show that any rotational hypersurface of type $\begin{cases} \text{I}, \\ \text{II}, \\ \text{III} \end{cases}$ carries a Kähler structure of quasi-constant holomorphic sectional curvatures with functions

$$\begin{cases} a + k^2 > 0, & a > 0, \\ a + k^2 > 0, & a < 0, \\ a + k^2 < 0, & a < 0, \end{cases} \quad \text{respectively.}$$

We describe the meridians of those rotational hypersurfaces, whose Kähler metrics are Bochner–Kähler (especially of constant holomorphic sectional curvatures).

6.1. Kähler structures on rotational hypersurfaces of type I

In [3] we studied the standard $2n$-dimensional rotational hypersurfaces M in $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ having no common points with the axis of revolution $l = \mathbb{R}$. Any such hypersurface M is a one-parameter family of spheres $S^{2n-1}(s),\ s \in I \subset \mathbb{R}$, considered as hyperspheres in $\mathbb{C}^n$ with corresponding centers on l and radii $t(s) > 0$, s being the natural parameter for the meridian. A rotational hypersurface M satisfying the conditions

$$t(s) > 0, \qquad t'(s) > 0; \quad s \in I$$

is said to be a *rotational hypersurface of type* I.

In [3] we have shown that any rotational hypersurface M of type I carries a natural Kähler structure (g, J, ξ), which has the following remarkable property.

Theorem 6.1 (*[3]*). *Let M* ($\dim M = 2n \geq 4$) *be a rotational hypersurface of type* I. *Then the Kähler structure (g, J, ξ) on M is of quasi-constant holomorphic sectional curvatures with functions*

$$a \geq 0, \quad (a + k^2 > 0).$$

The curvature tensor R of the metric g has the form

$$R = a\pi + b\Phi + c\Psi,$$

where

$$a = \frac{4(1-t')}{t^2}, \qquad b = 8\left(\frac{t'-1}{t^2} - \frac{t''}{2tt'}\right), \qquad c = \frac{4(1-t')}{t^2} + \frac{5t''}{2tt'} + \frac{t''^2 - t't'''}{2t'^3}. \tag{6.1}$$

In this subsection we describe the rotational hypersurfaces of type I, whose Kähler structure is Bochner flat.

We recall that the Bochner curvature tensor $B(R)$ of a Kähler manifold (M, g, J) ($\dim M = 2n \geq 4$) with curvature tensor R, Ricci tensor ρ and scalar curvature τ is given by

$$\begin{aligned}(B(R))_{\alpha\bar\beta\gamma\bar\delta} \ = \ & R_{\alpha\bar\beta\gamma\bar\delta} - \frac{1}{n+2}(g_{\alpha\bar\beta}\rho_{\gamma\bar\delta} + g_{\gamma\bar\beta}\rho_{\alpha\bar\delta} + g_{\gamma\bar\delta}\rho_{\alpha\bar\beta} + g_{\alpha\bar\delta}\rho_{\gamma\bar\beta}) \\ & + \frac{\tau}{2(n+1)(n+2)}(g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\gamma\bar\beta}g_{\alpha\bar\delta})\end{aligned} \tag{6.2}$$

in local holomorphic coordinates.

The manifold (M, g, J) is said to be Bochner flat (or the metric g is Bochner–Kähler) if $B(R) = 0$.

Lemma 6.2. *A Kähler manifold whose curvature tensor is of the form*

$$R = a\pi + b\Phi + c\Psi, \tag{6.3}$$

is Bochner flat if and only if $c = 0$.

Proof. Applying the Bochner operator (6.2) to the tensor (6.3) we find

$$B(R) = c\left(\frac{2}{(n+1)(n+2)}\pi - \frac{4}{n+2}\Phi + \Psi\right),$$

which gives the assertion. □

Any rotational hypersurface M of type I is geometrically determined by the equation $t = t(s)$ (or equivalently $s = s(t)$).

Proposition 6.3. *Let M be a rotational hypersurface of type* I. *Then the Kähler structure (g, J) is Bochner–Kähler if and only if*

$$s(t) = \int \frac{\mathrm{d}t}{c_1 t^4 + c_2 t^2 + 1},$$

where $c_1 = \text{const}$, $c_2 = \text{const}$.

Proof. Taking into account (6.1) we have

$$c = -\frac{t}{2t'}\left(\frac{t''}{tt'} + 4\frac{1-t'}{t^2}\right)'.$$

According to Lemma 6.2 we have to solve the equation $c = 0$, i.e.

$$\frac{t''}{tt'} + 4\frac{1-t'}{t^2} = \text{const} = -2c_2. \tag{6.4}$$

The general solution of (6.4) is

$$s(t) = \int \frac{\mathrm{d}t}{c_1 t^4 + c_2 t^2 + 1}$$

for some constant c_1. □

We note that the case $c_1 = 0$ gives the Kähler metrics of constant holomorphic sectional curvatures $a = \text{const} > 0$ described in [3].

6.2. Kähler structures on rotational hypersurfaces of type II

Let $(\mathbb{C}^n, g', J_0) = (\mathbb{R}^{2n}, g', J_0)$ be the complex space with the standard complex structure J_0 and flat definite metric g'. Further, let $O\mathbf{e}$ be a coordinate system on $\mathbb{R}$ with the inner product determined by $\mathbf{e}^2 = -1$ and $l = \mathbb{R}$ be the axis of revolution in the space $\mathbb{C}^n \times \mathbb{R}$. We denote the product metric in $\mathbb{R}^{2n}_1 = \mathbb{C}^n \times \mathbb{R}$ by the same letter g'. Then $g'(\mathbf{e}, \mathbf{e}) = -1$ and g' is of signature $(2n, 1)$.

We consider the class of rotational hypersurfaces having no common points with the axis of revolution l. Then any such hypersurface M is a one-parameter family of spheres $S^{2n-1}(s)$, $s \in I$ considered as hyperspheres in $\mathbb{C}^n$ with corresponding centers $q(s)\mathbf{e}$ on l and radii $t(s) > 0$. If $\mathbf{Z}$ is the radius vector of any point $p \in M$ with respect to the origin O, then the unit normal $\mathbf{n}$ of the parallel $S^{2n-1}(s)$ at the point p is

$$\mathbf{n} = \frac{\mathbf{Z} - q(s)\mathbf{e}}{t(s)}.$$

Hence

$$\mathbf{Z} = t(s)\mathbf{n} + q(s)\mathbf{e} \tag{6.5}$$

and the meridian γ of M is

$$\gamma : \mathbf{z}(s) = t(s)\mathbf{n} + q(s)\mathbf{e} \tag{6.6}$$

in the plane $O\mathbf{ne}$ ($\mathbf{n}$-fixed).

Because of (6.6) and (6.5) the tangent vector field $\bar{\xi}$ to γ is

$$\bar{\xi} = \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s} = t'\mathbf{n} + q'\mathbf{e} = \frac{\partial \mathbf{Z}}{\partial s}. \tag{6.7}$$

We consider rotational hypersurfaces whose meridian γ has a space-like tangent at any point and assume that s is a natural parameter for γ, i.e.

$$g'\left(\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s}\right) = t'^2 - q'^2 = 1.$$

Since the normal to M lies in the plane $O\mathbf{ne}$, we choose the time-like unit vector field N normal to M by the condition that the couples $(\mathbf{n}, \mathbf{e})$ and $(\bar{\xi}, N)$ have the same orientation. Then taking into account (6.7), we have

$$N = q'\mathbf{n} + t'\mathbf{e}.$$

Definition 6.4. A rotational hypersurface M in $\mathbb{R}^{2n}_1 = \mathbb{C}^n \times \mathbb{R}$, which has no common points with the axis of revolution l, is said to be *of type* II if its normals are time-like.

Let ∇' be the flat Levi-Civita connection of the metric g' in $\mathbb{R}^{2n}_1 = \mathbb{C}^n \times \mathbb{R}$. We denote the induced definite metric on M by $\bar{g}$. Let $\bar{\eta}$ be the 1-form corresponding to the space-like unit vector field $\bar{\xi}$ with respect to the metric $\bar{g}$, i.e. $\bar{\eta}(X) = \bar{g}(\bar{\xi}, X)$, $X \in \mathfrak{X}M$. If $\bar{\nabla}$ is the Levi-Civita connection on $(M, \bar{g})$, we have:

$$\begin{aligned} &\nabla'_X Y = \bar{\nabla}_X Y + \left(\frac{\sqrt{t'^2-1}}{t}\bar{g}(X,Y) + \frac{1-t'^2+tt''}{t\sqrt{t'^2-1}}\bar{\eta}(X)\bar{\eta}(Y)\right)N, \quad X, Y \in \mathfrak{X}M; \\ &\bar{\nabla}_{\bar{\xi}}\bar{\xi} = 0; \qquad \bar{\nabla}_x \bar{\xi} = \frac{t'}{t}x, \qquad \bar{g}(x,\bar{\xi}) = 0, \quad x \in \mathfrak{X}M. \end{aligned} \tag{6.8}$$

Then the curvature tensor $\bar{R}$ of the rotational hypersurface $(M, \bar{g})$ of type II has the form:

$$\bar{R} = -\frac{t'^2-1}{t^2}\,\bar{\pi} - \frac{1-t'^2+tt''}{t^2}\,\bar{\Phi}. \tag{6.9}$$

This equality implies that the rotational hypersurface $(M, \bar{g})$ of type II is conformally flat. More precisely, $(M, \bar{g}, \bar{\xi})$ is a subprojective Riemannian manifold with horizontal sectional curvatures $-\frac{t'^2-1}{t^2} \le 0$ (cf. [2]).

As in [3], we consider the almost contact Riemannian structure $(\varphi, \bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{g})$ on the parallels $S^{2n-1}(s)$, $s \in I$ of the rotational hypersurface M and obtain that any parallel is $\frac{1}{t}$-Sasakian.

This allows us to introduce the almost complex structure J on $(M, \bar{g})$ subordinated to the orientation $\bar{\xi}$ of the meridians by

$$J_{|D} := J_0, \qquad J\bar{\xi} := \bar{\bar{\xi}}, \qquad J\bar{\bar{\xi}} := -\bar{\xi}. \tag{6.10}$$

Similarly to the definite case [3] we have

Proposition 6.5. *Let $(M, \bar{g})$ be a rotational hypersurface of type* II *in $\mathbb{R}^{2n}_1 = \mathbb{C}^n \times \mathbb{R}$ whose meridians are oriented with the space-like unit vector field $\bar{\xi}$. If J is the almost complex structure (6.10) associated with $\bar{\xi}$, then the covariant derivative of J satisfies the identity*

$$(\bar{\nabla}_X J)Y = \frac{t'-1}{t}\left(\bar{g}(X,Y)\bar{\bar{\xi}} - \bar{\bar{\eta}}(Y)X - \bar{\eta}(Y)JX + \bar{g}(JX,Y)\bar{\xi}\right) \tag{6.11}$$

for all vector fields $X, Y \in \mathfrak{X}M$.

The identity (6.11) shows that $(M, \bar{g}, J)$ is a locally conformal Kähler manifold in all dimensions $2n \ge 4$ with Lee form $\frac{1-t'}{t}\,\bar{\eta}$.

Our aim in this subsection is to define a nontrivial Kähler metric on $(M, \bar{g}, J)$, which is naturally determined by its geometric structures.

If $(M, \bar{g}, J)$ is a rotational hypersurface of type II, then $t'^2 \ge 1$. Therefore we can always choose the orientation $\bar{\xi}$ of the meridians so that $t' \ge 1$.

In what follows we assume that

$$t(s) > 0, \qquad t'(s) \ge 1; \quad s \in I. \tag{6.12}$$

Under the conditions (6.12) we construct the structure (g, ξ):

$$g = \bar{g} + (t'-1)(\bar{\eta} \otimes \bar{\eta} + \bar{\bar{\eta}} \otimes \bar{\bar{\eta}}), \qquad \xi = \frac{1}{\sqrt{t'}}\,\bar{\xi}, \qquad \eta = \sqrt{t'}\,\bar{\eta}. \tag{6.13}$$

Taking into account (6.11) we obtain that the Kähler form of the metric (6.13) is closed, i.e. g is a Kähler metric. More precisely, we have

Theorem 6.6. *Let $(M, \bar{g}, J, \bar{\xi})$ $(2n \ge 4)$ be a rotational hypersurface of type* II *and assume that (6.12) holds good. Then the Kähler metric g, given by (6.13), is of quasi-constant holomorphic sectional curvatures with functions*

$$a \le 0, \quad a + k^2 > 0.$$

Proof. Calculating the relation between the connections of the metrics in (6.13) in view of (6.9) we find the curvature tensor R of the Kähler metric g:

$$R = a\pi + b\Phi + c\Psi,$$

where

$$a = \frac{4(1-t')}{t^2}, \qquad b = 8\left(\frac{t'-1}{t^2} - \frac{t''}{2tt'}\right), \qquad c = \frac{4(1-t')}{t^2} + \frac{5t''}{2tt'} + \frac{t''^2 - t't'''}{2t'^3}. \tag{6.14}$$

Applying Proposition 2.3 [3] we obtain that (M, g, J, ξ) is of quasi-constant holomorphic sectional curvatures.

Since $t' \geq 1$, then we have $a \leq 0$.

From (6.8) and the relation between the connections of g and $\bar{g}$ it follows that

$$\nabla_x \xi = \frac{\sqrt{t'}}{t} x - \frac{t''}{2t'\sqrt{t'}} \eta(Jx) J\xi$$

for all $x \in \mathfrak{X}M$, $g(\xi, x) = 0$. According to (2.1) the function k of the structure (g, J, ξ) is $k = 2\frac{\sqrt{t'}}{t}$. Taking into account (6.14) we find

$$a + k^2 = \frac{4}{t^2} > 0. \quad \square$$

As a consequence of Theorem 6.6 we can find the rotational hypersurfaces $(M, \bar{g}, J)$ of type II whose Kähler metric (6.13) is of constant holomorphic sectional curvatures.

Let $b = 0$ in (6.14). Then Corollary 3.6 [3] implies that $c = 0$ and the metric g is of constant holomorphic sectional curvatures $a = \text{const} \leq 0$.

Solving the equation

$$b = 8\left(\frac{t'-1}{t^2} - \frac{t''}{2tt'}\right) = 0,$$

we obtain the meridian in the form $q = q(t)$.

Namely, we have

Proposition 6.7. *Any rotational hypersurface* $(M, \bar{g}, J)$ *of type* II*, whose Kähler metric* (6.13) *is of constant holomorphic sectional curvatures* $a = \text{const} < 0$*, is generated by a meridian of the type*

$$\gamma : q = \pm\frac{1}{\sqrt{-a}}\left(\sqrt{8 - at^2} + \ln\frac{\sqrt{8-at^2} - 2}{\sqrt{8-at^2} + 2}\right) + q_0, \quad t > 0$$

in the hyperbolic plane O**ne**.

Similarly to Proposition 6.3 we obtain the following statement.

Proposition 6.8. *Let* $(M, \bar{g}, J)$ *be a rotational hypersurface of type* II *generated by the meridian*

$$\gamma : \mathbf{z}(s) = t(s)\mathbf{n} + q(s)\mathbf{e}, \quad s \in I$$

in the hyperbolic plane O**ne**. *Then the metric* g *given by* (6.13) *is Bochner–Kähler if and only if*

$$s(t) = \int \frac{\mathrm{d}t}{c_1 t^4 + c_2 t^2 + 1},$$

where $c_1 = \text{const}$, $c_2 = \text{const}$.

We note that the case $c_1 = 0$ is described in Proposition 6.7.

6.3. Kähler structures on rotational hypersurfaces of type III

Let $(\mathbb{C}^n, h', J_0) = (\mathbb{R}_2^{2(n-1)}, h', J_0)$ be the Kähler–Lorentz space with the standard complex structure J_0 and flat indefinite metric h' of signature $(2(n-1), 2)$. Further, let $O\mathbf{e}$ be a coordinate system on $\mathbb{R}$ with the inner product determined by $\mathbf{e}^2 = +1$ and $l = \mathbb{R}$ be the axis of revolution in the space $\mathbb{R}_2^{2(n-1)} \times \mathbb{R} = \mathbb{C}^n \times \mathbb{R}$. We denote the product metric in $\mathbb{R}_2^{2n-1} = \mathbb{C}^n \times \mathbb{R}$ by the same letter h'. Then $h'(\mathbf{e}, \mathbf{e}) = +1$ and h' is of signature $(2(n-1), 2)$.

We consider rotational hypersurfaces M with parallels $H_1^{2(n-1)}$, which are hyperspheres with respect to the metric h' in the time-like domain $\mathbb{T}_1^{n-1} \subset \mathbb{C}^n$. Then M is a one-parameter family of spheres $H_1^{2(n-1)}(s)$, $s \in I$ with corresponding centers $q(s)\mathbf{e}$ on l and radii $t(s) > 0$. If $\mathbf{Z}$ is the radius vector of any point $p \in M$ with respect to the origin O, then the unit normal $\mathbf{n}$ of the parallel $H_1^{2(n-1)}(s)$ at the point p is

$$\mathbf{n} = \frac{\mathbf{Z} - q(s)\mathbf{e}}{t(s)}, \quad h'(\mathbf{n}, \mathbf{n}) = -1.$$

Hence

$$\mathbf{Z} = t(s)\mathbf{n} + q(s)\mathbf{e} \tag{6.15}$$

and the meridian γ of M is

$$\gamma : \mathbf{z}(s) = t(s)\mathbf{n} + q(s)\mathbf{e} \tag{6.16}$$

in the plane $O\mathbf{ne}$ ($\mathbf{n}$- fixed).

Because of (6.16) and (6.15) the tangent vector field $\bar{\xi}$ to γ is

$$\bar{\xi} = \frac{d\mathbf{z}}{ds} = t'\mathbf{n} + q'\mathbf{e} = \frac{\partial \mathbf{Z}}{\partial s}. \tag{6.17}$$

We consider rotational hypersurfaces whose meridian γ has a time-like tangent at any point and assume that s is a natural parameter for γ, i.e.

$$h'\left(\frac{d\mathbf{z}}{ds}, \frac{d\mathbf{z}}{ds}\right) = -t'^2 + q'^2 = -1.$$

Since the normal to M lies in the plane $O\mathbf{ne}$, we choose the space-like unit vector field N normal to M by the condition that the couples $(\mathbf{n}, \mathbf{e})$ and $(\bar{\xi}, N)$ have the same orientation. Then taking into account (6.17), we have

$$N = q'\mathbf{n} + t'\mathbf{e}.$$

Definition 6.9. A rotational hypersurface M in $\mathbb{R}_2^{2n-1} = \mathbb{C}^n \times \mathbb{R}$, which has no common points with the axis of revolution $l = \mathbb{R}$, is said to be *of type* III if its normals are space-like.

Let ∇' be the flat Levi-Civita connection of the metric h' in $\mathbb{R}_2^{2n-1} = \mathbb{C}^n \times \mathbb{R}$. We denote by $\bar{h}$ the induced indefinite metric on M of signature $(2(n-1), 2)$. Let $\bar{\eta}$ be the 1-form corresponding to the unit time-like vector field $\bar{\xi}$ with respect to the metric $\bar{h}$, i.e. $\bar{\eta}(X) = \bar{h}(\bar{\xi}, X)$, $X \in \mathfrak{X}M$. If $\bar{\nabla}$ is the Levi-Civita connection on $(M, \bar{h})$, we have:

$$\begin{aligned} &\nabla'_X Y = \bar{\nabla}_X Y - \left(\frac{\sqrt{t'^2 - 1}}{t}\,\bar{h}(X, Y) + \frac{-1 + t'^2 + tt''}{t\sqrt{t'^2 - 1}}\bar{\eta}(X)\bar{\eta}(Y)\right) N, \quad X, Y \in \mathfrak{X}M; \\ &\bar{\nabla}_{\bar{\xi}}\bar{\xi} = 0; \qquad \bar{\nabla}_x \bar{\xi} = \frac{t'}{t}\,x, \qquad \bar{h}(x, \bar{\xi}) = 0, \quad x \in \mathfrak{X}M. \end{aligned} \tag{6.18}$$

Then the curvature tensor $\bar{R}$ of the rotational hypersurface $(M, \bar{h})$ of type III has the form:

$$\bar{R} = \frac{t'^2 - 1}{t^2}\,\bar{\pi} + \frac{-1 + t'^2 + tt''}{t^2}\,\bar{\Phi}, \tag{6.19}$$

where $\bar{\pi}$ and $\bar{\Phi}$ are the tensors

$$\bar{\pi}(X,Y)Z = \bar{h}(Y,Z)X - \bar{h}(X,Z)Y,$$
$$\bar{\Phi}(X,Y)Z = \bar{h}(Y,Z)\bar{\eta}(X)\bar{\xi} - \bar{h}(X,Z)\bar{\eta}(Y)\bar{\xi} + \bar{\eta}(Y)\bar{\eta}(Z)X - \bar{\eta}(X)\bar{\eta}(Z)Y, \quad X,Y,Z \in \mathfrak{X}M.$$

The equality (6.19) implies that the rotational hypersurface $(M,\bar{h})$ of type III is conformally flat.

Now we consider the almost contact Riemannian structure $(\varphi, \bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{h})$ on the parallel $H_1^{2(n-1)}(s),\ s \in I$ of the rotational hypersurface $(M,\bar{h})$ which arises in a similar way as in the definite case (cf. [8,9]):

$$\begin{aligned} &\bar{\bar{\xi}} := J_0 n, \qquad \bar{\bar{\eta}}(x) := \bar{h}(\bar{\bar{\xi}}, x);\\ &\varphi x := J_0 x - \bar{\bar{\eta}}(x) n, \quad x \in \mathfrak{X}H_1^{2(n-1)}(s). \end{aligned} \tag{6.20}$$

It is clear that $\bar{h}(\bar{\bar{\xi}}, \bar{\bar{\xi}}) = -1$. The relations (6.20) imply that

$$\varphi\xi = 0; \qquad \varphi^2 x = -x - \bar{\bar{\eta}}(x)\bar{\bar{\xi}};$$
$$\bar{h}(\varphi x, \varphi y) = \bar{h}(x,y) + \bar{\bar{\eta}}(x)\bar{\bar{\eta}}(y), \quad x,y \in \mathfrak{X}H_1^{2(n-1)}(s).$$

Let us denote by $\mathcal{D}$ the induced Levi-Civita connection of the metric $\bar{h}$ on $H_1^{2(n-1)}(s)$ as a submanifold of $\mathbb{T}_1^{n-1}(s) \subset \mathbb{C}^n$. Then the Weingarten and Gauss formulas of the imbedding $H_1^{2(n-1)}(s) \subset \mathbb{C}^n$ are:

$$\begin{aligned} &\nabla'_x \mathbf{n} = \frac{1}{t}\, x;\\ &\nabla'_x y = \mathcal{D}_x y + \frac{1}{t}\,\bar{h}(x,y)\mathbf{n}, \quad x,y \in \mathfrak{X}H_1^{2(n-1)}(s). \end{aligned} \tag{6.21}$$

From (6.20) and (6.21) we obtain consequently

$$\begin{aligned} &\nabla'_x \bar{\bar{\xi}} = \frac{1}{t}(\varphi x + \bar{\bar{\eta}}(x)\mathbf{n});\\ &\mathcal{D}_x \bar{\bar{\xi}} = \frac{1}{t}\varphi x, \quad x \in \mathfrak{X}H_1^{2(n-1)}(s). \end{aligned} \tag{6.22}$$

Let T_pM be the tangent space to M at any point $p \in M$. Then the vector fields $\bar{\xi}$ and $\bar{\bar{\xi}}$ defined by (6.20) determine a distribution D such that $D^\perp = \mathrm{span}\{\bar{\xi}, \bar{\bar{\xi}}\}$. The distribution D is space-like, while the distribution $D^\perp$ is time-like.

We define an almost complex structure J on $(M,\bar{h})$ subordinated to the orientation $\bar{\xi}$ of the meridians γ as follows:

$$J_{|D} := J_0, \qquad J\bar{\xi} := \bar{\bar{\xi}}, \qquad J\bar{\bar{\xi}} := -\bar{\xi}. \tag{6.23}$$

Similarly to Proposition 6.5 we have

Proposition 6.10. *Let $(M,\bar{h})$ be a rotational hypersurface of type* III *in $\mathbb{R}_2^{2n-1} = \mathbb{C}^n \times \mathbb{R}$ whose meridians γ are oriented with the time-like unit vector field $\bar{\xi}$. If J is the almost complex structure (6.23) associated with $\bar{\xi}$, then the covariant derivative of J satisfies the identity*

$$(\bar{\nabla}_X J)Y = \frac{1-t'}{t}\left(\bar{h}(X,Y)\bar{\bar{\xi}} - \bar{\bar{\eta}}(Y)X - \bar{\eta}(Y)JX + \bar{h}(JX,Y)\bar{\xi}\right) \tag{6.24}$$

for all vector fields $X, Y \in \mathfrak{X}M$.

Proof. We calculate the components of $(\bar{\nabla}_X J)Y$, $X, Y \in \mathfrak{X}M$:

$$(\bar{\nabla}_x J)y = \frac{1-t'}{t}(\bar{h}(x,y)\bar{\bar{\xi}} + \bar{h}(\varphi x, y)\bar{\xi} - \bar{\bar{\eta}}(y)n),$$
$$(\bar{\nabla}_x J)\bar{\xi} = \frac{1-t'}{t}\varphi x, \quad x,y \in \mathfrak{X}M, \qquad \bar{h}(\bar{\xi}, x) = \bar{h}(\bar{\xi}, y) = 0;$$

$$(\bar{\nabla}_{\bar{\xi}} J)x_0 = 0, \quad x_0 \in \mathfrak{X}M, \qquad \bar{h}(x_0, \bar{\xi}) = \bar{h}(x_0, \tilde{\bar{\xi}}) = 0;$$

$$(\bar{\nabla}_{\bar{\xi}} J)\tilde{\bar{\xi}} = 0, \qquad (\bar{\nabla}_{\bar{\xi}} J)\bar{\xi} = 0.$$

These equalities imply the assertion. □

The identity (6.24) shows that $(M, \bar{h}, J)$ is a locally conformal Kähler manifold in all dimensions $2n \geq 4$ with Lee form $\frac{1-t'}{t}\,\bar{\eta}$. This implies that $(M, \bar{h}, J)$ carries a conformal Kähler metric of signature $(2(n-1), 2)$ which is flat.

Our aim in this subsection is to define a nontrivial definite Kähler metric g on $(M, \bar{h}, J)$, which is naturally determined by its geometric structures.

If $(M, \bar{h}, J)$ is a rotational hypersurface of type III, then ${t'}^2 \geq 1$. Therefore we can always choose the orientation $\bar{\xi}$ of the meridians so that $t' \leq -1$.

In what follows we assume that

$$t(s) > 0, \qquad t'(s) \leq -1; \quad s \in I. \tag{6.25}$$

Under the conditions (6.25) we construct the structure (g, ξ):

$$g = \bar{h} + (1 - t')(\bar{\eta} \otimes \bar{\eta} + \tilde{\bar{\eta}} \otimes \tilde{\bar{\eta}}), \qquad \xi = \frac{1}{\sqrt{-t'}}\,\bar{\xi}, \qquad \eta = -\sqrt{-t'}\,\bar{\eta}. \tag{6.26}$$

Taking into account the defining condition (6.26) and (6.25), we obtain that g is a definite metric and ξ is a unit vector field. Because of (6.26) and (6.24) it follows that g is a Kähler metric on M.

More precisely, we have

Theorem 6.11. *Let $(M, \bar{h}, J, \bar{\xi})$ $(2n \geq 4)$ be a rotational hypersurface of type* III *and assume that (6.25) holds good. Then the Kähler metric g, given by (6.26), is of quasi-constant holomorphic sectional curvatures with functions*

$$a < 0, \quad a + k^2 < 0.$$

Proof. Let ∇ be the Levi-Civita connection of the metric (6.26). We calculate the relation between $\bar{\nabla}$ and ∇:

$$\begin{aligned}\bar{\nabla}_X Y = \nabla_X Y &- \frac{t''}{2t'\sqrt{-t'}}\{[\eta(X)\eta(Y) - \eta(JX)\eta(JY)]\xi - [\eta(X)\eta(JY) + \eta(JX)\eta(Y)]J\xi\} \\ &- \frac{1-t'}{t\sqrt{-t'}}\{\eta(JX)JY + \eta(JY)JX - [\eta(X)\eta(JY) + \eta(JX)\eta(Y)]J\xi \\ &- t'[g(X,Y) - \eta(JX)\eta(JY) - \eta(X)\eta(Y)]\xi - 2\eta(JX)\eta(JY)\xi\}\end{aligned}$$

for all $X, Y \in \mathfrak{X}M$.

Taking into account (6.18) we find

$$\nabla_X \xi = \frac{t'}{t\sqrt{-t'}}(X - \eta(X)\xi) - \frac{t''}{2t'\sqrt{-t'}}\eta(JX)J\xi, \quad X \in \mathfrak{X}M. \tag{6.27}$$

Then we find the curvature tensor R of the Kähler metric g:

$$R = a\pi + b\Phi + c\Psi,$$

where

$$a = \frac{4(t'-1)}{t^2}, \qquad b = -8\left(\frac{t'-1}{t^2} - \frac{t''}{2tt'}\right), \qquad c = \frac{4(t'-1)}{t^2} - 2\frac{t''}{tt'} - \frac{{t''}^2}{2t{t'}^3}\left(\frac{tt'}{t''}\right)'. \tag{6.28}$$

Applying Proposition 2.3 [3] we obtain that (M, g, J, ξ) is of quasi-constant holomorphic sectional curvatures.

Since $t' \leq -1$, then we have $a < 0$.

From (6.27) it follows that the function k of the structure (g, J, ξ) is $k = \frac{2t'}{t\sqrt{-t'}}$. Taking into account (6.28), we find

$$a + k^2 = -\frac{4}{t^2} < 0. \quad \square$$

As a consequence of Theorem 6.11 we can find the rotational hypersurfaces M of type III, whose Kähler metric (6.26) is of constant holomorphic sectional curvatures.

Let $b = 0$ in (6.28). Then Corollary 3.6 [3] implies that $c = 0$ and the metric g is of constant holomorphic sectional curvatures $a = \text{const} < 0$.

Solving the equation

$$b = -8\left(\frac{t' - 1}{t^2} - \frac{t''}{2tt'}\right) = 0,$$

we obtain the meridian in the form $q = q(t)$.

Namely, we have

Proposition 6.12. *Any rotational hypersurface* $(M, \bar{h}, J)$ *of type* III*, whose Kähler metric* (6.28) *is of constant holomorphic sectional curvatures* $a = \text{const} < 0$*, is generated by a meridian of the type*

$$\gamma : q = \frac{1}{-a}\left(\sqrt{a(8 + at^2)} - 2\sqrt{-a}\ \arctan\frac{1}{2}\sqrt{-(8 + at^2)}\right), \quad t > \frac{2\sqrt{2}}{\sqrt{-a}}$$

in the hyperbolic plane O**ne**.

Similarly to Propositions 6.3 and 6.8 we obtain the following statement.

Proposition 6.13. *Let* $(M, \bar{h}, J)$ *be a rotational hypersurface of type* III *generated by the meridian*

$$\gamma : \mathbf{z}(s) = t(s)\mathbf{n} + q(s)\mathbf{e}, \quad s \in I$$

in the hyperbolic plane O**ne**. *Then the metric* g *given by* (6.28) *is Bochner–Kähler if and only if*

$$s(t) = \int \frac{\mathrm{d}t}{c_1 t^4 + c_2 t^2 + 1},$$

where $c_1 = \text{const}$, $c_2 = \text{const}$.

We note that the case $c_1 = 0$ is described in Proposition 6.12.

References

[1] R. Bryant, Bochner–Kähler metrics, J. Amer. Math. Soc. 14 (2001) 623–715.
[2] G. Ganchev, V. Mihova, Riemannian manifolds of quasi-constant sectional curvature, J. Reine Angew. Math. 522 (2000) 119–141.
[3] G. Ganchev, V. Mihova, Kähler manifolds of quasi-constant holomorphic sectional curvatures, Arxiv:math.DG/0505671 (in press).
[4] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, Kodai Math. J. 4 (1981) 1–27.
[5] K. Ogiue, On almost contact manifolds admitting axiom of planes or axiom of free mobility, Kodai Math. Sem. Rep. 16 (1964) 223–232.
[6] S. Tachibana, R.C. Liu, Notes on Kählerian metrics with vanishing Bochner curvature tensor, Kodai Math. Sem. Rep. 22 (1970) 313–321.
[7] S. Tanno, Sasakian manifolds with constant φ-holomorphic sectional curvature, Tôhoku Math. J. 21 (1969) 501–507.
[8] Y. Tashiro, On contact structures on hypersurfaces in almost complex manifolds I, Tôhoku Math. J. 15 (1963) 62–79.
[9] Y. Tashiro, On contact structures on hypersurfaces in almost complex manifolds II, Tôhoku Math. J. 15 (1963) 167–175.
[10] F. Tricerri, L. Vanhecke, Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc. 267 (1981) 365–398.